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A New Maximum Principle for Impulsive First-Order Problems

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We prove a new maximum principle for a boundary value problem for first-order ordinary differential equations with impulses at fixed moments.

1. INTRODUCTION

It is often convenient in mechanics and in other branches of science to consider an idealized extreme in which a force is applied for only an infinitesimal amount of time, but still communicates a nonzero quantity of momentum. This leads to the concepts of impulse function and impulsive differential equation.

Maximum principles play a central role in the theory of differential equations. They are used to study qualitative aspects such as existence, uniqueness of solutions, multiplicity of solutions, stability, and order of convergence of numerical schemes.

Differential equations with impulses are a basic tool to study evolution processes that are subject to abrupt changes in their states. For instance, many biological phenomena involving thresholds or optimal control models in economics exhibit impulsive effects (Lakshmikantham *et al.*, 1989; Samoilenko and Perestyuk, 1995). Hence it is of the utmost importance to develop a general theory for differential equations including some basic aspects of this theory.

In this paper we study a linear impulsive differential equation and present a new maximum principle which generalize previous known results.

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2. PRELIMINARIES

We consider the following linear problem:

$$(LP)\quad \begin{cases} u'(t) + \lambda u(t) = \sigma(t), & t \neq t_k, \quad t \in J = [0, T] \\ u(t_k^+) = c_k u(t_k) \\ u(0) = u(T) + \mu \end{cases}$$

where $\lambda, \mu \in \mathbb{R}$, $0 = t_0 < t_1 < \ldots < t_p < t_{p+1} = T$, $c_k \in \mathbb{R}$, $k = 1, \ldots, p$, and $\sigma: J \to \mathbb{R}$ is such that σ is continuous for $t \neq t_k$, there exist the limits $\sigma(t_k^-) = \lim_{h\to 0^-} \sigma(t_k + h)$, $\sigma(t_k^+) = \lim_{h\to 0^+} \sigma(t_k + h)$, and $\sigma(t_k^-) = \sigma(t_k)$ for every $k = 1, \ldots, p$.

In order to define more precisely the concept of solution for the problem (LP), we introduce the following spaces of functions:

$$PC(J) = \{u: J \to \mathbb{R};\ u_{|(t_k, t_{k+1})} \in C((t_k, t_{k+1})),\ k = 0, 1, \ldots, p,$$

$$\exists u(0^+),\ u(T^-),\ u(t_k^+),\ u(t_k^-), \text{ and } u(t_k^-) = u(t_k),\ k = 1, \ldots, p\}$$

and

$$PC^1(J) = \{u \in PC(J); u_{|(t_k, t_{k+1})} \in C^1((t_k, t_{k+1})),\ k = 0, 1, \ldots, p,$$

$$\exists u'(0^+),\ u'(T^-),\ u'(t_k^+), \text{ and } u'(t_k^-),\ k = 1, \ldots, p\}$$

$PC(J)$ and $PC^1(J)$ are Banach spaces with the norms

$$\|u\|_{PC(J)} = \sup\{|u(t)|;\ t \in J\}$$

and

$$\|u\|_{PC^1(J)} = \|u\|_{PC(J)} + \|u'\|_{PC(J)}$$

Note that

$$\|u\|_{PC(J)} = \sup\{\|u_k\|_{C(J_k)};\ k = 0, 1, \ldots, p\},\ u_k = u_{|J_k},\ J_k = [t_k, t_{k+1}]$$

and, in this sense, $PC(J)$ is equivalent to $\prod_{k=0}^{p} C(J_k)$.

By a solution of problem (LP) we mean a function $u \in PC^1(J)$ satisfying the conditions indicated in (LP).

In the nonimpulsive case, that is, if $c_k = 1$, $k = 1, \ldots, p$, then $u(t_k^+) = u(t_k)$ and $u \in C[0, T]$. It is well known that (LP), has a unique solution for any $\sigma \in C(J)$ and $\mu \in \mathbb{R}$ if and only if $\lambda \neq 0$.

In the case $\lambda = 0$, the problem is solvable if and only if $\int_0^T \sigma(s)ds = 0$. In such a case, there is an infinite number of solutions. In other words, the only eigenvalue of u' with periodic conditions is $\lambda = 0$ (the eigenfunctions being constants). We have the following maximum principle depending on the sign of $\lambda \neq 0$.

Theorem 2.1. Consider the problem (*LP*) with $\sigma \in C(J)$ and $C_\kappa = 1$, $k = 1, \ldots, p$:

1. If $\sigma \geq 0$ in J and $\mu \geq 0$, then

$$\begin{cases} \lambda > 0 \Rightarrow u \geq 0 \\ \lambda < 0 \Rightarrow u \leq 0 \end{cases}$$

2. If $\sigma \leq 0$ in J and $\mu \leq 0$, then

$$\begin{cases} \lambda > 0 \Rightarrow u \leq 0 \\ \lambda < 0 \Rightarrow u \geq 0 \end{cases}$$

This maximum principle is essential for developing the monotone iterative method, a powerful theorical method, (Ladde *et al.*, 1985), which permits us to construct a sequence of approximate solutions converging to a solution of the nonlinear problem

$$\begin{cases} u'(t) = f(t, u(t)), & t \in J \\ u(0) = u(T) \end{cases}$$

In the impulsive case, (LP) is not always solvable (even if $\lambda \neq 0$). See the examples in (Nieto, 1997). The next result gives precise information on the eigenvalues in the impulsive case. It also gives an explicit formula for the solution.

Theorem 2.2. (LP) has a unique solution for all $\sigma \in PC(J)$ if and only if $\prod_{k=1}^{p} c_k \neq e^{\lambda T}$.

Moreover, in this case and for $\lambda \neq 0$ the solution satisfies the following equation for every $t \in [0, T]$:

$$u(t) = \int_0^T g(t, s)\sigma(s)\, ds + \sum_{j=1}^{p} g(t, t_j)(c_j - 1)u(t_j) + \frac{\mu e^{-\lambda t}}{1 - e^{-\lambda T}} \quad (2.1)$$

where

$$g(t, s) = \frac{1}{1 - e^{-\lambda T}} \begin{cases} e^{-\lambda(t-s)} & \text{if} \quad 0 \leq s < t \leq T \\ e^{-\lambda(T+t-s)} & \text{if} \quad 0 \leq t \leq s \leq T \end{cases} \quad (2.2)$$

Proof. Let $u(0) = u_0$. With this initial condition and the two first equations of (LP) we have a Cauchy problem which is solvable and it has a unique solution u for each u_0.

Using the variation of parameters theorem for systems with impulses (Theorem 2.5.1 in Lakshmikantham, *et al.*, 1989), it is easy set up the following expression for $u(t)$, $t \in J$:

$$u(t) = e^{-\lambda t}u(0) + \int_0^t e^{-\lambda(t-s)}\sigma(s)\,ds + \sum_{\{k:0<t_k<t\}} e^{-\lambda(t-t_k)}(c_k - 1)u(t_k) \quad (2.3)$$

In particular, for $t = T$ we have

$$u(T) = u(0)\prod_{k=1}^{p} c_k e^{-\lambda T} + \int_0^{t_1}\prod_{k=1}^{p} c_k e^{-\lambda(T-s)}\sigma(s)\,ds + \int_{t_1}^{t_2}\prod_{k=2}^{p} c_k e^{-\lambda(T-s)}\sigma(s)\,ds + \ldots + \int_{t_p}^{T} e^{-\lambda(T-s)}\sigma(s)\,ds \quad (2.4)$$

Now, a solution of the Cauchy problem will be a solution of (LP) if and only if satisfies the boundary condition $u(0) = u(T) + \mu$. Then, by (2.4),

$$\left(1 - \prod_{k=1}^{p} c_k e^{-\lambda T}\right)u(0) = \mu + \int_0^{t_1}\prod_{k=1}^{p} c_k e^{-\lambda(T-s)}\sigma(s)\,ds + \int_{t_1}^{t_2}\prod_{k=2}^{p} c_k e^{-\lambda(T-s)}\sigma(s)\,ds + \ldots + \int_{t_p}^{T} e^{-\lambda(T-s)}\sigma(s)\,ds$$

Thus, for every $\sigma \in PC(J)$ there exists an initial condition $u(0)$ satisfying the boundary condition if and only if

$$\prod_{k=1}^{p} c_k \neq e^{\lambda T}$$

Now, for $\lambda \neq 0$ we have

$$u(0) = \frac{\mu}{1 - e^{-\lambda T}} + \int_0^T \frac{e^{-\lambda(T-s)}\sigma(s)}{1 - e^{-\lambda T}}\,ds + \sum_{k=1}^{p} \frac{e^{-\lambda(T-t_k)}}{1 - e^{-\lambda T}}(c_k - 1)u(t_k)$$

and substituting this value into (2.3), we obtain the relation (2.1). ■

Remark 2.1. For $\lambda \neq 0$ and $\mu = 0$ it is proved in Franco (1997) and Nieto (1997) that $u \in PC^1(J)$ is a solution of (LP) if and only if u satisfies (2.1).

Theorem 2.1 was partially generalized to equations with impulses in Lakshmikantham *et al.* (1989).

Theorem 2.3. Consider the problem (LP) with $\lambda > 0$, $\sigma \geq 0$ in J, $c_k > 0$, $k = 1, 2, \ldots, p$, $\mu \geq 0$, and $\prod_{k=1}^{p} c_k e^{-\lambda T} < 1$. Then $u(t) \geq 0$, $t \in J$.

But in this result there is no information about the sign of u when either $\lambda < 0$ or $\prod_{k=1}^{p} c_k e^{-\lambda T} < 1$.

We now define the operator L: $D(L) \to PC(J)$ by

$$L(u)(t) = u'(t),\ t \neq t_k, \quad L(u)(t_k) = u'(t_k^-), \qquad k = 1, \ldots, p$$

where

$$D(L) = \{u \in PC^1(J) \colon u(t_k^+) = c_k u(t_k),\ k = 1, \ldots, p,\ u(0) = u(T)\}$$

We observe that the problem (LP) with $\mu = 0$ is equivalent to the abstract equation

$$Lu = \sigma, \qquad u \in D(L)$$

Theorem 2.3 can be seen as a sufficient condition to assure that the operator $(L + \lambda I)$ is inverse positive, that is, $(L + \lambda I)(u) \geq 0$ on J implies that $u \geq 0$ on J.

Now, we introduce some definitions and results (Berman *et al.*, 1990) that we will be useful in the following sections.

Definition 2.1. A real $p \times p$ matrix $Q = (q_{ji})$ is said to be monotone if $Qx \geq 0$ implies $x \geq 0$, $x \in \mathbb{R}^p$.

Proposition 2.1 Let $Q = (q_{ji}) \in M_{p\times p}(\mathbb{R})$ such that $q_{ji} \leq 0$, $\forall i \neq j$. Then the following two conditions are equivalent:

1. Q is monotone.
2. Every principal minor of Q is positive.

The following result gives us an important inequality for piecewise continuous functions. It is a particular case of Corollary 1.4.1 in Lakshmikantham *et al.* (1989).

Lemma 2.1. Let $s \in J$ and let u, $a \in PC^1(J)$, $\lambda \in \mathbb{R}$, and $c_k \geq 0$, $k = 1, \ldots, p$, constants such that

(*i*) $u'(t) \leq a(t)u(t) \qquad t \in [s, T), \quad t \neq t_k$

$u(t_k^+) \leq c_k u(t_k), \qquad t_k \in [s, T)$

Then we have for $t \in [s, T)$

$$u(t) \leq u(s^+) \prod_{s<t_k<t} c_k \exp\left[\int_s^t a(z)dz\right] \tag{2.5}$$

(*ii*) $u'(t) \geq a(t)u(t) \qquad t \in [s, T), \quad t \neq t_k$

$u(t_k^+) \geq c_k u(t_k) \qquad t_k \in [s, T)$

Then we have for $t \in [s, T)$ the reverse inequality of (2.5),

$$u(t) \geq u(s^+) \prod_{s<t_k<t} c_k \exp\left[\int_s^t a(z)\,dz\right]$$

3. MAXIMUM PRINCIPLES VIA MONOTONE MATRICES

In this section we explain how to use monotone matrices to obtain a maximum principle for (LP).

Lemma 3.1. Consider the problem (LP) with $\sigma(t) \geq 0$, $t \in J$, and $\mu \geq 0$. If $\lambda > 0$, and u is a solution of (LP) such that $u(t_j^+)$ and $u(t_j^-)$ are nonnegative for $j = 1, \ldots, p$, then $u \geq 0$ in J.

Proof. Let $s \in J$ such that

$$u(s) = \min_{t \in J} \{u(t)\}$$

Suppose that $u(s) < 0$ (obviously $s \neq t_k$, $k = 1, \ldots, p$). We first suppose that $s = T$. Then there exists $s_1 \in [t_p, T)$ such that $u(s_1^+) = 0$, $u(t) < 0$, $t \in (s_1, T]$. The mean value theorem implies that there exists $s_2 \in (s_1, T)$ such that $u'(s_2) < 0$ and $u(s_2) < 0$. But in this situation we obtain the following contradiction:

$$0 > u'(s_2) + \lambda u(s_2) = \sigma(s_2) \geq 0$$

Now, if $s = 0$, then $s = T$ since $\mu \geq 0$. If $s \in int(J)$, then $u'(s) = 0$, and

$$0 \leq u'(s) + \lambda u(s) < 0$$

which is impossible. ∎

Remark 3.1. Lemma 3.1 is a maximum principle, but it is not practical since it is necessary to know *a priori* the values of the function u at some fixed instants $t_1, t_2, \ldots, t_p$. On the other hand, if $c_k > 0$, to guarantee that $u(t_k^+)$ and $u(t_k^-)$ are nonnegative for $k = 1, \ldots, p$ is sufficient that $u(t_k^-)$ are nonnegative for $k = 1, 2, \ldots, p$.

Suppose that $\lambda > 0$, $\mu \geq 0$, and $\sigma \geq 0$. By Theorem 2.2 we know that

$$u(t) - \sum_{i=1}^{p} g(t, t_i)(c_i - 1)u(t_i) \geq 0, \qquad t \in J$$

For the impulsive instants we obtain the following p inequalities:

$$u(t) - \sum_{i=1}^{p} g(t_i, t_j)(c_j - 1)u(t_j) \geq 0, \qquad i = 1, \ldots, p \tag{3.1}$$

We define the vector $u = (u(t_1), \ldots, u(t_p))^T \in \mathbb{R}^p$ and the $p \times p$ matrix $Q = (q_{ij})$ by

$$q_{ij} = \frac{1}{1 - e^{-\lambda T}} \begin{cases} -e^{-\lambda(T+t_i-t_j)}\,(c_j - 1) & \text{if } i > j \\ 1 - (c_j - 1)e^{-\lambda T} & \text{if } i = j \\ -e^{-\lambda(t_i-t_j)}(c_j - 1) & \text{if } i < j \end{cases}$$

Note that off-diagonal entries of Q are nonpositive and therefore Q is under the conditions of Proposition 2.1.

With this notation, it is clear that (3.1) can be written as

$$Q\tilde{u} \geq 0$$

If we prove that Q is monotone, then we have that $\tilde{u} = (u(t_1), \ldots, u(t_p))^T \geq 0$ and we can apply Lemma 3.1.

It is easily seen that the $s \times s$ principal minor of Q is

$$M_s(Q) = \frac{1}{1 - e^{-\lambda T}} \left(1 - \prod_{k=1}^{s} c_k e^{-\lambda T}\right), \qquad s = 1, 2, \ldots, p$$

It is positive if and only if $\prod_{k=1}^{s} c_k < e^{\lambda T}$ since $\lambda > 0$. Now, suppose that $c_k > 1$, $k = 1, \ldots, p$. It follows that all the principal minors of Q are positive if and only if $\det(Q) = M_p(Q) > 0$, and this is equivalent to the condition

$$\prod_{k=1}^{p} c_k < e^{\lambda T}$$

Now, we can use Lemma 3.1 to obtain that $u(t) \geq 0$, $t \in J$. Thus, we have just proved the following result.

Theorem 3.1. Consider the problem (*LP*) with $\lambda > 0$, $\sigma \geq 0$ in J, $c_k > 1$, $k = 1, 2, \ldots, p$, $\mu \leq 0$, and $\prod_{k=1}^{p} c_k e^{-\lambda T} > 1$. Then $u(t) \geq 0$, $t \in J$.

Remark 3.2. Note that the sign of λ is a crucial factor to prove the result of Theorem 3.1. If one tries to generalize this result to the case $\lambda < 0$, then it is not possible to employ a similar method.

The result of Theorem 3.1 is weaker than the conclusion of Theorem 2.3 due to the conditions imposed on the constants c_k, $k = 1, \ldots, p$. We include it in this paper because with it, it is easy to see that, for $\lambda > 0$, the condition $\prod_{k=1}^{p} c_k e^{-\lambda T} < 1$ is also necessary for the operator $(L + \lambda I)$ to be inverse positive (see Theorem 3.1 in Liz and Nieto, 1998).

Theorem 3.2. Let $\lambda > 0$. The operator $(\mathrm{L} + \lambda I)$ is inverse positive in $D(L)$ if and only if $\prod_{k=1}^{p} c_k e^{-\lambda T} < 1$.

In view of this and Theorem 2.2, it is clear that the meaningful factor to set up the sign of a solution of (LP) is the sign of $\prod_{k=1}^{p} c_k e^{-\lambda T} - 1$.

4. MAIN RESULT

We present a new maximum principle for (LP) which depends only on the sign of

$$\prod_{k=1}^{p} c_k e^{-\lambda T} - 1 \tag{4.1}$$

Note that in this case the sign of λ is not important (compare with Theorems 2.1 and 3.1).

Theorem 4.1. Consider the problem (LP) with $c_k > 0$, $k = 1, \ldots, p$:

1. If $\sigma \geq 0$, and $\mu \geq 0$, then

$$\begin{cases} \prod_{k=1}^{p} c_k e^{-\lambda T} < 1 \Rightarrow u \geq 0 \\ \prod_{k=1}^{p} c_k e^{-\lambda T} > 1 \Rightarrow u \geq 0 \end{cases}$$

2. If $\sigma \leq 0$, and $\mu \leq 0$, then

$$\begin{cases} \prod_{k=1}^{p} c_k e^{-\lambda T} < 1 \Rightarrow u \leq 0 \\ \prod_{k=1}^{p} c_k e^{-\lambda T} > 1 \Rightarrow u \geq 0 \end{cases}$$

Proof. We present only the proof of part 1, since the justification of part 2 is analogous.

By Lemma 2.1, if $u \in PC^1(J)$ is solution of (LP), then u satisfies

$$u(t) \geq u(0) \prod_{\{k: t_k < t\}} c_k e^{-\lambda t}, \qquad \forall t \in J \tag{4.2}$$

We contemplate two different cases depending on the sign of (4.1):

(i) $\prod_{k=1}^{p} c_k e^{-\lambda T} < 1$. By (4.2), it is sufficient to prove that $u(0) \geq 0$. If $u(0) < 0$, we obtain that

$$u(0) \geq u(T) \geq u(0) \prod_{k=1}^{p} c_k e^{-\lambda T} > u(0)$$

which is a contradiction.

(ii) $\prod_{k=1}^{p} c_k e^{-\lambda T} > 1$. Hence, $u(0) \leq 0$.

To see that $u \leq 0$ on J, suppose that there exists $s \in J$ such that $u(s) > 0$. If $s = t_j$ for some $j \in \{1, \ldots, p\}$, then $u(s^-) > 0$ and $u(s^+) > 0$. By Lemma 2.1, it follows that

$$u(t) \geq u(s) \prod_{\{k: s < t_k < t\}} c_k e^{-\lambda(t-s)}, \qquad \forall t > s$$

In particular, for $t = T$ we have

$$u(T) \geq u(s) \prod_{\{k: s < t_k\}} c_k e^{-\lambda(T-s)} > 0$$

This proves that $u(T) > 0$, and in consequence,

$$u(T) > u(0)$$

which is imposible since $\mu \leq 0$. ■

We obtain the following consequence, which generalizes Theorem 3.2

Corollary 4.1 The operator $(L + \lambda I)$ is inverse positive in $D(L)$ if and only if $\prod_{k=1}^{p} c_k e^{-\lambda T} < 1$.

It is easy to generalize Theorem 4.1 to the following problem

$$(P) \quad \begin{cases} u'(t) + \lambda u(t) = \sigma(t), & t \neq t_k, t \in J = [0, T] \\ u(t_k^+) = I_k(u(t_k)) \\ u(0) = u(T) + \mu \end{cases}$$

where $\lambda, \mu \in \mathbb{R}$, $0 = t_0 < t_1 < \ldots < t_p < t_{p+1} = T$, and $I_k: \mathbb{R} \to \mathbb{R}$, $k = 1, \ldots, p$, satisfy that there exist constants $c_k > 0$, $k = 1, \ldots, p$, such that either

$$I_k(x) \geq c_k x, \qquad x \in \mathbb{R} \tag{4.3}$$

or

$$I_k(x) \leq c_k x, \qquad x \in \mathbb{R} \tag{4.4}$$

Thus, we have

Theorem 4.2. Consider the problem (P):

1. If $\sigma \geq 0$, $\mu \geq 0$, and (4.3) holds, then

$$\begin{cases} \prod_{k=1}^{p} c_k e^{-\lambda T} < 1 \Rightarrow u \geq 0 \\ \prod_{k=1}^{p} c_k e^{-\lambda T} > 1 \Rightarrow u \leq 0 \end{cases}$$

2. If $\sigma \leq 0$, $\mu \leq 0$, and (4.4) holds, then

$$\begin{cases} \prod_{k=1}^{p} c_k e^{-\lambda T} < 1 \Rightarrow u \leq 0 \\ \prod_{k=1}^{p} c_k e^{-\lambda T} > 1 \Rightarrow u \geq 0 \end{cases}$$

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REFERENCES

Berman, A., Neumann, M., and Stern, R. J. (1990). *Nonnegative Matrices in Dynamic Systems*, Wiley, New York.

Franco, D. (1997). *Periodic solutions of impulsive differential equations*, Master dissertation, Departamento Análisis Matemático, Universidad de Santiago de Compostela, Spain.

Ladde, G. S., Lakshmikantham, V., and Vatsala, A. S. (1985). *Monotone Iterative Techniques for Nonlinear Differential Equations*, Pitman, Boston.

Lakshmikantham, V., Bainov, D. D., and Simeonov, P. S. (1989). *Theory of Impulsive Differential Equations*, World Scientific, Singapore.

Liz, E., and Nieto, J. J. (1998). *Positive solutions of linear impulsive differential equations*, *Communications in Applied Analysis,* **2**, 565–572.

Nieto, J. J. (1997). Basic theory for nonresonance impulsive periodic problems of first order, *Journal of Mathematical Analysis and Applications,* **205**, 423–433.

Samoilenko, A. M., and Perestyuk, N. A. (1995). *Impulsive Differential Equations*, World Scientific, Singapore.